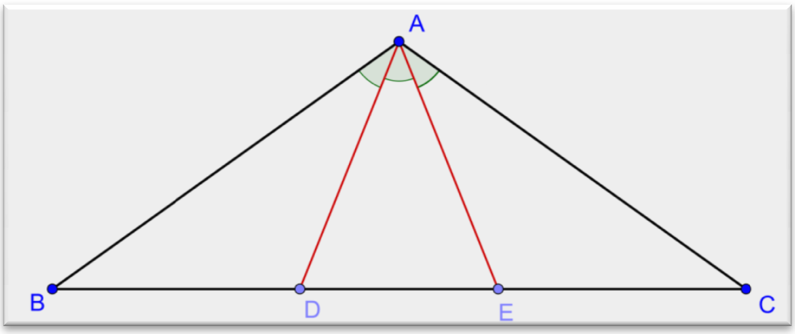
Find a length

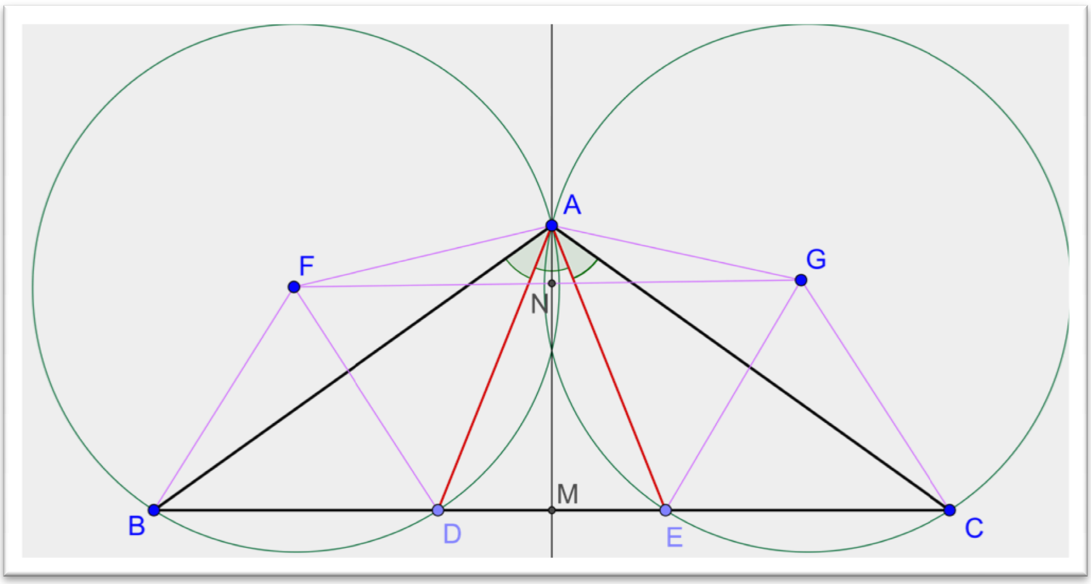
In $\triangle$ ABC, D and E are two points on BC such that BD < BE, BD = NE = 5 and DE = 4. If $\angle$BAD = $\angle$DAE = $\angle$EAC, find the length of AC.



Solution

First we have to prove AB = AC and AD = AE.

By sine law, for any triangle ABC, we have $\frac{a}{\sin A} = 2R$, where R is the circum-radius of the circum-circle passing through a triangle ABC.



Now the circum-radius, R_1, of ΔBDA $= \frac{BD}{2 \sin BAD}$

And the circum-radius, R_2, of ΔCEA $= \frac{CE}{2 \sin CAE}$

Hence $R_1 = R_2$

Then FA = FB = FD = GA = GC = GE

Draw $AN \perp FG$ and produce to meet BC at M.

$\Delta ANF \cong \Delta ANG$, RHS

$FN = NG$, corr. Sides of congruent Δs

So we have both circum-centres are symmetric with respect to the perpendicular bisector of FG, which must then contain A.

The diagram is then symmetric with respect to the line AM and AM is the perpendicular bisector of BC
Therefore AB=AC and AD = AE.

Since AD, AE bisect $\angle BAE$, $\angle CAD$ respectively, we get

$$\frac{\text{Area of } \Delta ABD}{\text{Area of } \Delta AED} = \frac{\frac{1}{2} AB \times AE \sin BAD}{\frac{1}{2} AE \times AE \sin BAE} = \frac{4}{5}$$

$$\therefore \frac{AE}{AB} = \frac{4}{5}$$

Similarly $\frac{AD}{AC} = \frac{4}{5}$

$$\therefore \frac{AE}{AB} = \frac{4}{5} = \frac{AD}{AC}$$

Let $x = AC = AB$

$$AD = \frac{4}{5} x = AE$$

By cosine law on ΔDAE, $3^2 = \left(\frac{4}{5}x\right)^2 + \left(\frac{4}{5}x\right)^2 - 2\left(\frac{4}{5}x\right)\left(\frac{4}{5}x\right) \cos DAE$

By cosine law on ΔEAC, $5^2 = \left(\frac{4}{5}x\right)^2 + x^2 - 2\left(\frac{4}{5}x\right) x \cos EAC$

Since $\angle DAE = \angle EAC$,

$$\frac{\left(\frac{4}{5}x\right)^2 + \left(\frac{4}{5}x\right)^2 - 9}{2\left(\frac{4}{5}x\right)\left(\frac{4}{5}x\right)} = \frac{\left(\frac{4}{5}x\right)^2 + x^2 - 25}{2\left(\frac{4}{5}x\right)x}$$

$$\frac{4}{5}\left[\left(\frac{4}{5}x\right)^2 + x^2 - 25\right] = \left(\frac{4}{5}x\right)^2 + \left(\frac{4}{5}x\right)^2 - 9$$

$$\frac{4x^2}{125} - 11 = 0$$

$$\therefore x = \frac{5\sqrt{55}}{2} \quad \text{(taking positive root)}$$

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